Diophantine equations
and when to quit trying to solve them

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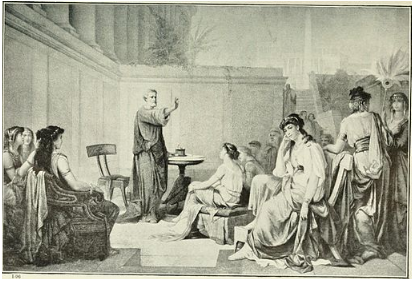
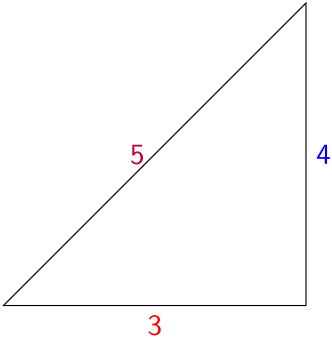
Diophantine equations

Mathematicians working on Diophantine equations study the integer solutions to polynomial equations with integer coefficients.

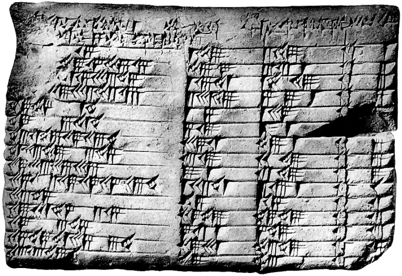


E.g. the Pythagorean equation $a^2 + b^2 = c^2$ has the integer solution $a = 3, b = 4, c = 5$.



$$3^2 + 4^2 = 5^2$$

Plimpton 322 (c. 1800 BC)



$$
\begin{aligned}
119^2 + 120^2 &= 169^2 \\
3367^2 + 3456^2 &= 4825^2 \\
4601^2 + 4800^2 &= 6649^2 \\
12709^2 + 13500^2 &= 18541^2
\end{aligned}
$$

Rational points on algebraic varieties

Let $f(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$.

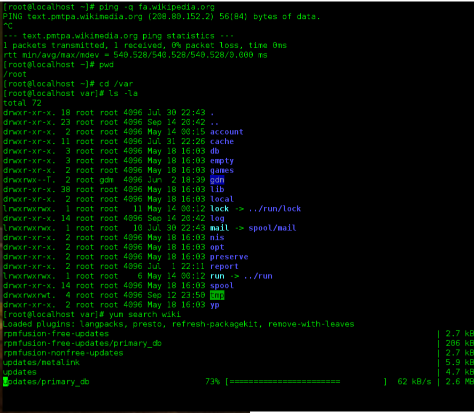
Let $X/\mathbb{Q}$ be the variety defined by $f(x_1, \ldots, x_n) = 0$.

The set of **rational points** on X is

$$X(\mathbb{Q}) = \{(x_1, \ldots, x_n) \in \mathbb{Q}^n \mid f(x_1, \ldots, x_n) = 0\}.$$

E.g. $(0, 1)$ is a rational point on the unit circle $x^2 + y^2 - 1 = 0$.

Searching for rational points



Using $\mathbb{R}$ to prove that no rational points exist

Let $X/\mathbb{Q}$ be an algebraic variety.

$$X(\mathbb{Q}) \subset X(\mathbb{R})$$

so

$$X(\mathbb{R}) = \emptyset \implies X(\mathbb{Q}) = \emptyset.$$

$X(\mathbb{R})$ is easier to deal with than $X(\mathbb{Q})$ because $\mathbb{R}$ is complete.

The real world is not enough

But

$$X(\mathbb{R}) \neq \emptyset \not\Rightarrow X(\mathbb{Q}) \neq \emptyset.$$

E.g. $x^2 = 2$ has real solutions but no rational solutions.

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$$\mathbb{R} = \frac{\{\text{Cauchy sequences in } \mathbb{Q} \text{ with respect to } |\cdot|\}}{\{\text{sequences in } \mathbb{Q} \text{ converging to } 0 \text{ with respect to } |\cdot|\}}$$

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E.g. π can be represented by the Cauchy sequence

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots$$

or by

$$3 + 1, 3.1 + 1/2, 3.14 + 1/3, 3.141 + 1/4, 3.1415 + 1/5, 3.14159 + 1/6, \ldots$$

Entering the p-adic world

Let p be a prime. Define the p-adic absolute value $|\cdot|_p$ on $\mathbb{Q}$ by

$$\left|p^r \frac{a}{b}\right|_p = p^{-r}$$

where $r, a, b \in \mathbb{Z}$ and $p \nmid a, b$. We also set $|0|_p = 0$.

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$1, 11, 111, 1111, 11111, \cdots \to -1/9$ with respect to $|\cdot|_5$.

The p-adic numbers

$$\mathbb{Q}_p = \frac{\{\text{Cauchy sequences in } \mathbb{Q} \text{ with respect to } |\cdot|_p\}}{\{\text{sequences in } \mathbb{Q} \text{ converging to } 0 \text{ with respect to } |\cdot|_p\}}$$

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i.e. $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Concretely, elements of $\mathbb{Q}_p$ look like

$$\sum_{n=N}^{\infty} a_n p^n, \qquad a_n \in \{0, 1, \ldots, p-1\}, \ N \in \mathbb{Z}.$$

The Hasse principle

$$X(\mathbb{Q}) \subset X(\mathbb{R}) \times \prod_p X(\mathbb{Q}_p) = X(\mathbb{A}_\mathbb{Q})$$

$$Q \mapsto (Q, Q, Q, Q, Q, \dots).$$

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Definition

If " $\Longleftarrow$ " holds, we say the **Hasse principle** holds.

The Hasse–Minkowski Theorem

Theorem (Hasse–Minkowski)

The Hasse principle holds for quadratic forms. I.e. a quadratic form over $\mathbb{Q}$ has a non-trivial zero over $\mathbb{Q}$ iff it has non-trivial zeros over $\mathbb{R}$ and over $\mathbb{Q}_p$ for all primes p.

A counterexample to the Hasse principle

Example (Lind, Reichardt)

The curve

$$C : 2y^2 = x^4 - 17z^4$$

has points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for all p but no points over $\mathbb{Q}$.

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How do we prove that $C(\mathbb{Q}) = \emptyset$?

We need a tool that combines information at different primes.

Hilbert symbols

Let $a, b \in \mathbb{Q}_p \setminus \{0\}$. Define the Hilbert symbol $(a, b)_p$ as follows:

$$(a, b)_p = \begin{cases} 1/2 & \text{if } as^2 + bt^2 = u^2 \text{ has no nontrivial solution over } \mathbb{Q}_p; \\ 0 & \text{otherwise.} \end{cases}$$

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Similarly, for $a, b \in \mathbb{R} \setminus \{0\}$, write

$$(a,b)_\infty = \begin{cases} 1/2 & \text{if } as^2 + bt^2 = u^2 \text{ has no nontrivial solution over } \mathbb{R}; \\ 0 & \text{otherwise.} \end{cases}$$

Hilbert symbols

Theorem (equivalent to quadratic reciprocity)

Let $a, b \in \mathbb{Q} \setminus \{0\}$*. Then*

$$\sum_{p \leq \infty} (a, b)_p \in \mathbb{Z}.$$

Returning to Lind and Reichardt's counterexample

Recall the curve

$$C : 2y^2 = x^4 - 17z^4.$$

The symbol $(y, 17)$ can be evaluated at a $\mathbb{Q}_p$-point or an $\mathbb{R}$-point of C to give a Hilbert symbol.

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E.g. let $Q = (\sqrt{2}, \sqrt{2}, 0) \in C(\mathbb{R})$. Then

$$(y, 17)(Q) = (y_Q, 17)_\infty = (\sqrt{2}, 17)_\infty.$$

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Note that

$$\sqrt{2}s^2 + 17t^2 = u^2$$

has the solution $(0, 1, \sqrt{17})$ over $\mathbb{R}$. Therefore, $(\sqrt{2}, 17)_\infty = 0$.

Returning to Lind and Reichardt's counterexample

One can show that for any $Q \in C(\mathbb{Q})$, we have

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E.g. $\sqrt{17} \in \mathbb{Q}_{13}$ so

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On the other hand, we will show that $(y_Q, 17)_{17} = 1/2$. This gives

$$\sum_{p \leq \infty} (y_Q, 17)_p = 1/2 \notin \mathbb{Z},$$

contradicting the reciprocity theorem. This implies that $C(\mathbb{Q}) = \emptyset$.

Proving that $(y_Q, 17)_{17} = 1/2$

Let $Q = (x_Q, y_Q, z_Q) \in C(\mathbb{Q})$. So

$$2y_Q^2 = x_Q^4 - 17z_Q^4. \qquad (1)$$

Rescaling, we can assume that $x_Q, y_Q, z_Q \in \mathbb{Z}$.

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Suppose for contradiction that

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has a nontrivial solution over $\mathbb{Q}_{17}$. Then $y_Q \equiv (u/s)^2 \pmod{17}$.

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Plugging this into (1) modulo 17 gives

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Hence 2 is a 4th power modulo 17. This is a contradiction because the only 4th powers modulo 17 are $0, \pm 1, \pm 4$. So $(y_Q, 17)_{17} = 1/2$. □

The Brauer group

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Let X be a nice variety and let $\mathcal{A} \in \mathrm{Br}\, X$. For all $p \leq \infty$, we have evaluation maps

$$X(\mathbb{Q}_p) \times \mathrm{Br}\, X \to \mathbb{Q}/\mathbb{Z}$$
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Recalling that $X(\mathbb{A}_\mathbb{Q}) = \prod_{p \leq \infty} X(\mathbb{Q}_p)$, we obtain a pairing

$$\begin{aligned} X(\mathbb{A}_\mathbb{Q}) \times \mathrm{Br}\, X &\to \mathbb{Q}/\mathbb{Z} \\ ((Q_p)_{p \leq \infty}, \mathcal{A}) &\mapsto \sum_{p \leq \infty} \mathcal{A}(Q_p). \end{aligned}$$

Key observation (Manin, 1970):

$$X(\mathbb{Q}) \subset \left\{ (Q_p)_{p\leq\infty} \in X(\mathbb{A}_{\mathbb{Q}}) \mid \forall \mathcal{A} \in \operatorname{Br} X,\ \sum_{p\leq\infty} \mathcal{A}(Q_p) = 0 \in \mathbb{Q}/\mathbb{Z} \right\}.$$

The set on the right-hand side is the Brauer–Manin set, denoted $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}$.

Brauer–Manin obstructions

Key observation (Manin, 1970):

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- Suppose $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$ but $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$. Then $X(\mathbb{Q}) = \emptyset$.

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  Brauer–Manin obstruction to the Hasse principle

Computing the Brauer–Manin set

To compute $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}}$, for each $\mathcal{A} \in \mathrm{Br}\, X$ we need to understand how $\mathcal{A}(Q_p)$ varies as Q_p varies in $X(\mathbb{Q}_p)$.

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For example, if $\mathcal{A}$ has order n then $\mathcal{A}(Q_p) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

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For example, if $\mathcal{A}$ has order n then $\mathcal{A}(Q_p) \in \frac{1}{n}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$.

If for some p all values in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ are attained then $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} \neq \emptyset$, i.e. $\mathcal{A}$ **does not obstruct the Hasse principle**.

Computing the Brauer–Manin set

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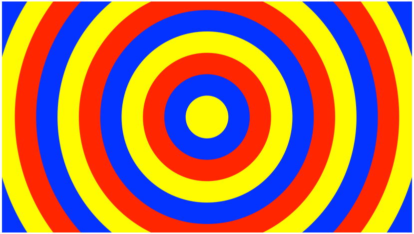
- If $\mathcal{A}$ has order coprime to p then $\mathcal{A}(Q_p)$ only depends on $Q_p \bmod p$.
- If $\mathcal{A}$ has order p then $\mathcal{A}(Q_p)$ could depend on $Q_p \bmod p^2$ or $\bmod p^3$ etc.

Wild evaluation maps



Bright–N., 2020

For $\mathcal{A} \in \operatorname{Br} X$ of order p, we:

- calculate m such that $\mathcal{A}(Q_p)$ only depends on Q_p mod p^m
- show that $\mathcal{A}(Q_p)$ varies linearly on discs of points that are the same mod p^{m-1}...
- ...unless $p \mid m$, when variation can be quadratic

Which primes can be involved in the Brauer–Manin obstruction?

Let $\mathcal{A} \in \operatorname{Br} X$.

Question (Swinnerton-Dyer, 2010)

Suppose that $\operatorname{Pic} \bar{X}$ is torsion-free. Let p be a prime of good reduction for X (i.e. X mod p is smooth). Is $\mathcal{A}(Q_p)$ constant as Q_p varies in $X(\mathbb{Q}_p)$?

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Equivalently, let $S = \{\text{primes of bad reduction}\} \cup \{\infty\}$. Does

$$X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = Z \times \prod_{p \notin S} X(\mathbb{Q}_p),$$

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Does the Brauer–Manin obstruction involve only primes of bad reduction and infinite primes?

Which primes can be involved in the Brauer–Manin obstruction?

Theorem (Bright–N., 2020)

If $\mathrm{H}^0(X, \Omega_X^2) \neq 0$ *then every prime of good ordinary reduction is involved in a Brauer–Manin obstruction over some extension of the base field.*

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Theorem (Margherita Pagano, 2021)

Let

$$X : x^3y + y^3z + z^3w + w^3x + xyzw = 0$$

and let $\mathcal{A} = \left(\frac{z^3+w^2x+xyz}{x^3}, \frac{-z}{x}\right) \in \mathrm{Br}\, X$. *Then 2 is a prime of good reduction for* X *and* $\mathcal{A}(Q_2)$ *is not constant as* Q_2 *varies in* $X(\mathbb{Q}_2)$.

Career overview

2021 – present Reader in Number Theory at KCL

2016 – 2021 Lecturer then Assoc. Prof. at Reading

2012 – 2015 Postdoc at Leiden, MPIM Bonn, IHÉS

2008 – 2012 PhD Cambridge

2007 – 2008 Part III Cambridge

2004 – 2007 BSc Warwick

Some things I learnt during my career

- Research is the primary criterion

  (for academic teaching and research jobs)
- Be strategic re teaching experience
  - Check out the Nesin Maths Village!
- Give good, comprehensible talks and lots of them. Ask for honest feedback.
  - Any talk in Germany is potentially a job talk (this may also apply elsewhere)
- Talk to big shots at conferences

Some more things I learnt during my career

- The sniper versus the scattergun approach to job applications
  - Write to people you want to work with, even if nothing is advertised
  - Don't waste time on pointless applications
- Play the long game
  - Prestige can be a means to an end
- Money matters
  - When interviewing for permanent/tenure track jobs, find out about the funding landscape in that country so you can talk about the grants you intend to apply for if you get the job